Lect26-0421 Pi-1 Examples

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For a space X with $x_0 \in X$, we have

$$\pi_1(X, x_0) = \{\text{loops in } X \text{ at } x_0\} / \text{loop homotopy}$$

Will show later to be independent of choice of x_0 if X is path connected.

It is a group with the multiplication

$$[\alpha]\cdot[\beta] = [\underbrace{\alpha * \beta}]$$

First walk α then walk β.

Examples

① $X = \mathbb{R}^n$, $n \geq 1$; or $X \subset \mathbb{R}^n$ is star-shaped.

* Every map into $X \simeq$ constant map
* $\pi_1(X) = \{1\}$

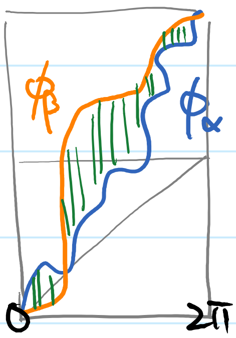
② $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$, punctured plane

* Every loop class $[\alpha] \longmapsto$ winding number $\in \mathbb{Z}$

  onto is easy

* One-to-one, i.e.,

  same winding number for $[\alpha], [\beta]$ $\Longrightarrow$ $[\alpha] = [\beta]$ i.e. loop homotopic



$$\pi_1(\mathbb{C} \setminus \{0\}) = (\mathbb{Z}, +)$$

Circle, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$

Consider $S^1 \xhookrightarrow{\iota} \mathbb{C}\setminus\{0\} \xrightarrow{r} S^1$

$z \longmapsto z \qquad \qquad \frac{z}{|z|}$

This gives homomorphisms

$$\pi_1(S^1) \xrightarrow{\iota_\#} \pi_1(\mathbb{C}\setminus\{0\}) \xrightarrow{r_\#} \pi_1(S^1)$$

Here is a situation called deformation retract.

We have

$$\begin{tikzcd} S^1 \arrow[r, hook, "\iota"] \arrow[rr, bend left, "r\circ\iota = id_{S^1}"] & \mathbb{C}\setminus\{0\} \arrow[r, "r"] \arrow[rr, bend right, "\iota\circ r \simeq id_{\mathbb{C}\setminus\{0\}}"'] & S^1 \arrow[r, hook, "\iota"] & \mathbb{C}\setminus\{0\} \end{tikzcd}$$

On the fundamental groups, it becomes

$r_\# \circ \iota_\# = id : \pi_1(S^1) \longrightarrow \pi_1(S^1)$ and

$\iota_\# \circ r_\# = id : \pi_1(\mathbb{C}\setminus\{0\}) \longrightarrow \pi_1(\mathbb{C}\setminus\{0\})$

So, we have $\iota_\#$ is 1-1 and $r_\#$ is onto

and $r_\#$ is 1-1 and $\iota_\#$ is onto

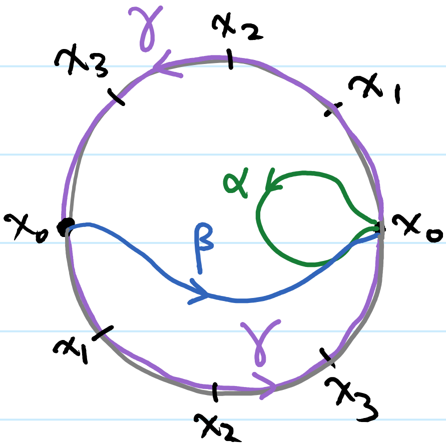
Therefore,
$$\pi_1(S^1) \underset{r_\#}{\overset{\iota_\#}{\rightleftarrows}} \pi_1(\mathbb{C}\setminus\{0\}) = (\mathbb{Z}, +)$$

shows that $\pi_1(S^1)$ is isomorphic to $(\mathbb{Z}, +)$.

Projective Plane, $\mathbb{RP}^2$

$\mathbb{RP}^2 = \mathbb{D}/\sim$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$

and $\sim$ identifies antipodal points on the boundary of $\mathbb{D}$.



In the picture, we have

$[\alpha] = 1$

$[\beta] \neq 1$

"Pushing" β downward,

$[\beta] = [\gamma]$

Note that $[\gamma]$ can be represented by either the lower half or upper half semi-circle on the boundary

"Pushing" β upward, $[\beta] = [\gamma]^{-1}$

So, we have $[\gamma] = [\gamma]^{-1}$ or $[\gamma]^2 = 1$

$\pi_1(\mathbb{RP}^2)$ is a cyclic group of order 2

$\cong (\mathbb{Z}/2, +)$

+	0	1
0	0	1
1	1	0

Torus

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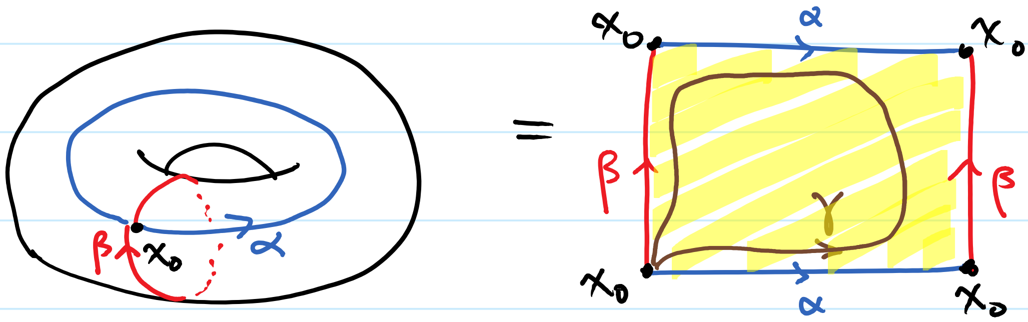
Torus $= S^1 \times S^1$

Theorem. Let X, Y be path connected and $x_0 \in X$, $y_0 \in Y$. Then
$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Hence, $\pi_1(\text{Torus}) = \pi_1(S^1 \times S^1)$
$$= \pi_1(S^1) \times \pi_1(S^1) = (\mathbb{Z}, +) \times (\mathbb{Z}, +) = \mathbb{Z} \oplus \mathbb{Z}$$

Torus as a quotient space



$$[\alpha] \longmapsto (1, 0)$$
$$[\beta] \longmapsto (0, 1)$$

Moreover, the loop $\alpha * \beta * \bar{\alpha} * \bar{\beta} \simeq c$,
$$\therefore [\alpha][\beta] = [\beta][\alpha]$$

Or, $[\alpha][\beta][\alpha]^{-1}[\beta]^{-1} = [\gamma]$ and $\gamma \simeq c$

Every loop based at x_0 on the torus

$\simeq$ a product of α and β

$= [\alpha]^m \cdot [\beta]^n \longmapsto (m, n)$

$\therefore \pi_1(\text{torus}) = (\mathbb{Z} \oplus \mathbb{Z}, +)$

Atypical ones

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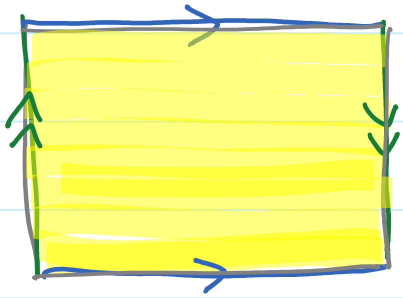
Klein Bottle. It is obtained by the quotient.

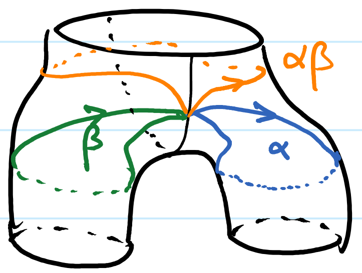


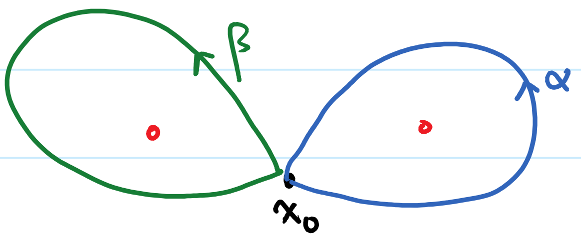
Combining the argument for $\mathbb{RP}^2$ and Torus,

$\pi_1(\text{Klein}) = (\mathbb{Z} \oplus \mathbb{Z}/_2, +)$

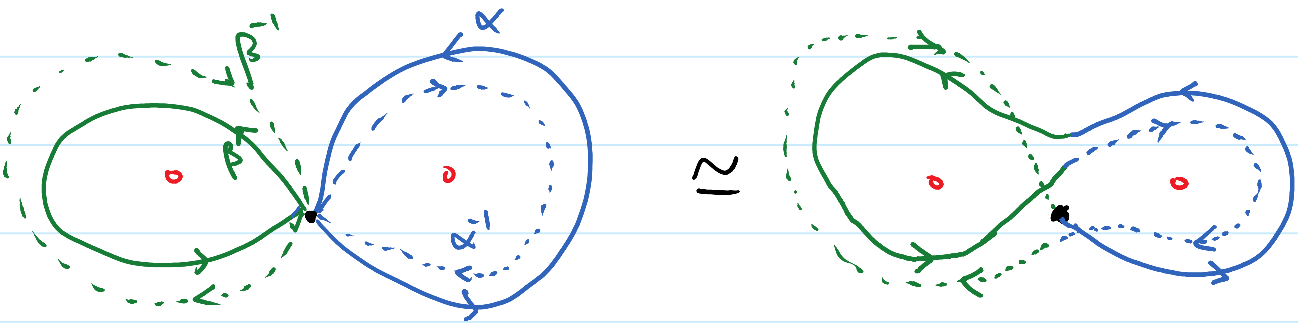
Two punctured plane, or pair of pants

$\mathbb{C} \setminus \{2 \text{ pts}\} = S^2 \setminus \{3 \text{ pts}\} =$





Consider what will $\alpha\beta\alpha^{-1}\beta^{-1}$ be. Note that up to loop homotopy, we may move the loop with only the start and end fixed.



$\alpha\beta\alpha^{-1}\beta^{-1} \neq 1$

Others

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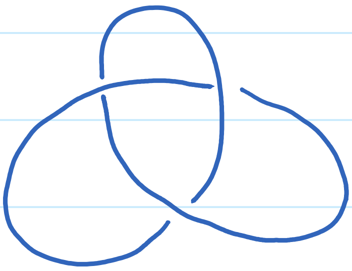
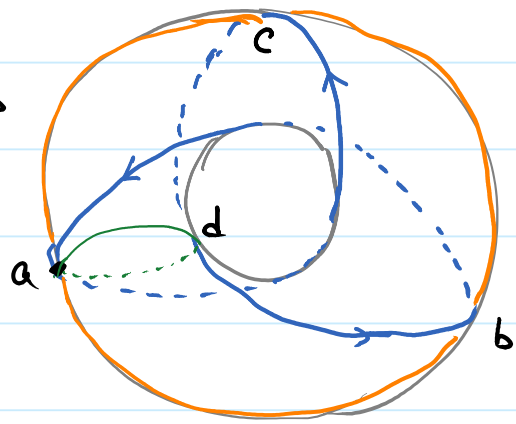
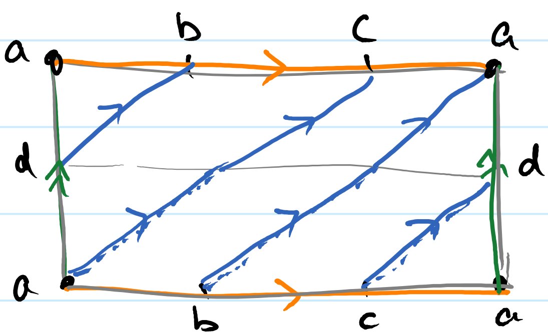
Torus knot (Digression)



Trefoil knot can be placed on a torus

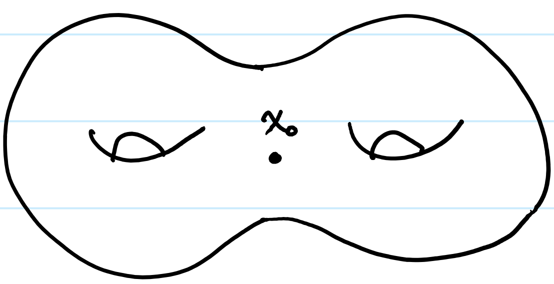




This knot goes 3-times in ○-direction a 2-times in ○-direction.

$\therefore$ [trefoil] $\in \pi_1(\text{Torus}) \longmapsto (3,2) \in \mathbb{Z} \oplus \mathbb{Z}$

Example 4 Surface of genus g



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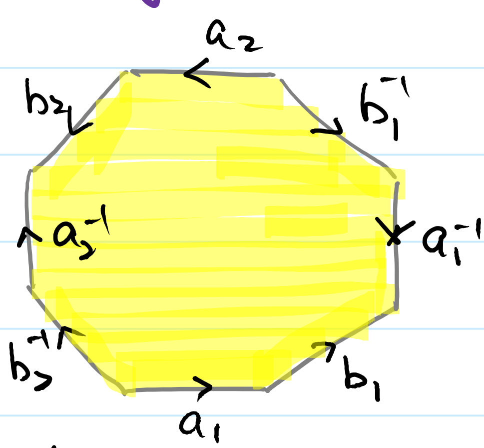

In general,

$$\pi_1\begin{pmatrix}\text{Surface}\\ \text{genus } g\end{pmatrix} = \left\langle a_1, b_1, \cdots, a_g, b_g : a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \cdots\cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \right\rangle$$

A group with $4g$ generators and one relation